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# More 3-manifolds with multiple knot-surgery and branched-cover descriptions

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For any integer $N \geqslant 3$ we will construct a 3-manifold which can be described as $+1$ surgery on $N$ distinct knots in $S^3$. We will also give examples of 3-manifolds which are $N$-fold cyclic branched covers of $S^3$ over 2 distinct knots. Brakes (2) discovered the first examples of 3-manifolds with multiple knot surgery descriptions. Our construction is much different and follows directly from the construction used by Lickorish (6) to describe a manifold which has 2 distinct knot surgery descriptions. Giller (5) has given examples of 3-manifolds which arise as cyclic branched covers over distinct knots in $S^3$. Our construction is similar, but the knots are much easier to distinguish, being iterated torus knots.

Throughout this paper we will be working in the smooth category.

## 1. *Manifolds with multiple knot-surgery descriptions*

A *Brunnian* link is a link in $S^3$ with the property that every proper sublink is trivial. Let $L = L_1 \cup L_2 \cup \ldots \cup L_n$ be a Brunnian link of $n$ components. If $+1$ surgery is performed on all the components of $L$ other than $L_j$, the resulting manifold is $S^3$ (as surgery on a trivial link gives $S^3$) and $L_j$ represents some knot in $S^3$, which we denote by $L_j^*$. Hence, the manifold obtained as $+1$ surgery on all the components of $L$, denoted by $M_L$, can be described as surgery on any of the knots $L_i^*$, $i = 1, \ldots, n$. If $n \geqslant 3$ the surgery coefficients will be $+1$, as all linking numbers in $L$ will be 0. Our object is to construct a link $L$ such that the $L_i^*$ are distinct.

If $K$ is an oriented knot in $S^3$, the $(1,p)$-*cable* about $K$ is the knot represented by a simple closed curve on the torus boundary of a tubular neighbourhood of $K$, where the curve is homologous to $p$ times $K$ in that tubular neighbourhood and is homologous to a positively oriented meridian to $K$ in $S^3 - K$. (This curve does not depend on the orientation of $K$.)

As the $(1,p)$-cable of the unknot is unknotted, if any component of a Brunnian link is replaced by the $(1,p)$-cable about that component, the link is again Brunnian.

LEMMA. *Let $L$ be a Brunnian link, and $\hat{L}$ be the Brunnian link obtained by replacing $L_j$ with the $(1,p)$-cable about $L_j$, $p \geqslant 3$. If $M_L$ is not simply connected then $M_{\hat{L}}$ is not simply connected.*

*Proof.* As $M_L$ is not simply connected and is given by $+1$ surgery on $L_j^*$, it is clear that $L_j^*$ is nontrivial. $\hat{L}_j^*$ is the $(1,p)$-cable of $L_j^*$. According to (9) surgery on a cable knot of this type $(p \geqslant 3)$ always results in a non-simply connected manifold. ∎

*Example.* To construct an example of a manifold with $n$ surgery descriptions, start with the Brunnian link $L$ of Figure 1, first described by Debrunner (3). The manifold $M_L$ is not simply connected. To see this, notice that there is an obvious $Z_n$ action on $M_L$ with fixed point set a circle. (Rotate Figure 1 through an angle of $2\pi/n$ about the centre.) The quotient space is $+1$ surgery on the square knot. This is not simply connected, by the proof of ((1), Theorem 5·1). However, the quotient of a homotopy sphere by a cyclic action with a circle of fixed points is always simply connected. (Any closed loop in the quotient is homotopic to a loop that lifts to a closed curve, via a homotopy possibly intersecting the branch set. That lift bounds a disk, so its projection bounds a disk also. Hence, the original loop was null homotopic.)

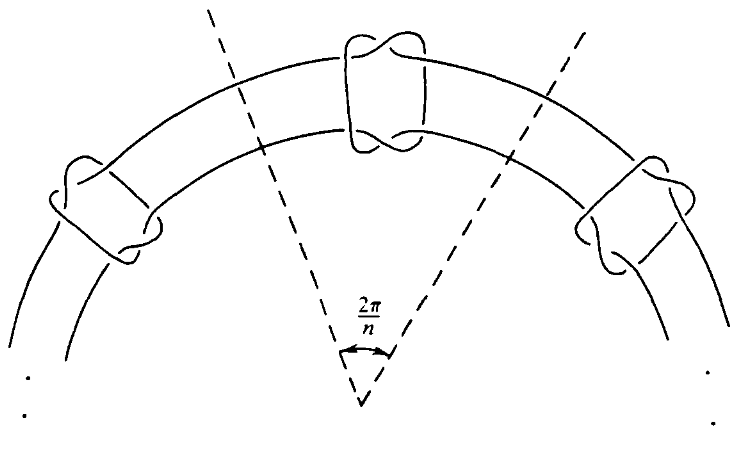


Fig. 1

Form a new link $\bar{L}$ by replacing each component of $L$, $L_j$, with its $(1, j+2)$-cable. $\bar{L}_j^*$ is hence the $(1, j+2)$-cable about some knot. By the Lemma, $M_{\bar{L}}$ is not simple connected, so $\bar{L}_j^*$ is a cable about a nontrivial knot. According to a result of Schubert (8) (see also Fox ((4), p. 144)) cables of nontrivial knots are distinguished by their cabling numbers. Hence, the $\bar{L}_j^*$ are distinct.

## 2. *Manifolds with multiple branched-cover descriptions*

Giller (5) used the following procedure to construct 2 distinct knots in $S^3$ with the same $n$-fold cyclic branched covers. Let $L$ be an oriented link of 2 components in $S^3$,

$$\begin{tikzcd} & N_L \arrow[dl, "q_1"'] \arrow[dd, "p"] \arrow[dr, "q_2"] & \\ N_{K_1} \arrow[dr, "p_1"'] & & N_{K_2} \arrow[dl, "p_2"] \\ & S^3 & \end{tikzcd}$$

Fig. 2

$K_1$ and $K_2$. There is a natural $Z_n \oplus Z_n$ branched cover of $S_3$ coming from the map $H_1(S^3 - L; Z) \to Z_n \oplus Z_n$ sending positively oriented meridians of $K_1$ and $K_2$ to $(1, 0)$ and $(0, 1)$ respectively. Call that cover $N_L$. Using the projection maps onto the first or second summand of $Z_n \oplus Z_n$, the branched cover $N_L \xrightarrow{p} S^3$ can be factored in two ways. Here $p_1$ and $p_2$ are the $n$-fold cyclic branched covers of $S^3$ over $K_1$ and $K_2$; $q_1$ and $q_2$ are the $n$-fold cyclic branched covers of $N_{K_1}$ and $N_{K_2}$ over the lifts of $K_2$ and $K_1$, respectively (see Figure 2).

If $K_1$ and $K_2$ are unknotted, then $N_{K_1}$ and $N_{K_2}$ are both $S^3$. If $lk(K_1, K_2)$ is prime to $n$, then the lifts of $K_2$ and $K_1$ to $N_{K_1}$ and $N_{K_2}$ are connected. If both these conditions hold, then $N_L$ is the branched cover of $S^3$ in two ways. The object is to arrange $L$ so that the lifts of $K_1$ and $K_2$ are distinct.

Here is our example of distinct knots with the same $r$-fold cyclic branched covers. Pick $p$ and $q$ relatively prime to $r$, and distinct. Start with a Hopf link. (A $(2, 2)$-torus link.) Replace one component with its $(1, p)$-cable, the other by its $(1, q)$-cable, to form a link $L$. The two conditions described in the preceding paragraph are met. It is a straightforward exercise to show that one lift is the $(r, q)$-cable of a $(r, p)$-torus knot, the other is the $(r, p)$-cable of the $(r, q)$-torus knot. These are distinct, again using Schubert (8).

## 3. *Comments*

Cabling can be used to construct other examples as well. For instance, $+1$ surgery could be replaced by $1/n$ surgery throughout Section 1. If one starts with the Hopf link and follows this construction, the conclusion is that certain surgeries on iterated torus knots can give the same manifold. This appears also in (2). If $n \geqslant 4$, the knots $L_i^*$ cannot be distinguished by their Alexander polynomials. In fact, it is an easy exercise using Rolfsen's description of the Alexander polynomial (7) to show the $L_i^*$ all have polynomial 1. It is interesting that Lickorish referred to (7) as a simple means of distinguishing his examples (6).

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